Approximation by Complex Szász-Mirakyan-Stancu-Durrmeyer Operators in Compact Disks under Exponential Growth

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Abstract. In the present paper, we deal with the complex Szász-Stancu-Durrmeyer operators and study Voronovskaja type results with quantitative estimates for these operators attached to analytic functions of exponential growth on compact disks. Also, the exact order of approximation is found.

1. Introduction

In 1985, Mazhar and Totik [23], studied the approximation properties in the real domain for the Szász-Durrmeyer operators. Then, Gupta and Agrawal [15] estimated the rate of convergence for functions of bounded variation. Also, very recently, approximation properties for several real operators including the Szász-Durrmeyer operators are presented in the papers [1], [22], [25], [28] and in the book Gupta and Agarwal [14].

In the complex domain, the overconvergence phenomenon holds, that is the extension of approximation properties from real domain to complex domain. In this context, the first qualitative kind results were obtained in the papers [4], [30], [31]. Then, in the books of Gal [6], [9] quantitative approximation results are presented for several type of approximation operators. For Szász-Mirakjan operator and its Stancu variant in complex domain, we refer the readers to [2], [3], [17], [5], [19], [20], [29] and [16]. Also for complex Bernstein-Durrmeyer operators, several papers are available in the literature (see e. g. [7], [8], [11], [12], [13], [21], [24], [26], [27]), for complex Szász-Durrmeyer operators see [10], while for complex q-Balász-Szabados operators see [18].

In this paper, we extend the studies and we discuss approximation properties of the Stancu variant for the complex Szász-Durrmeyer operator given (for $0 \leq \alpha \leq \beta$) by

$$S_n^{(\alpha,\beta)}(f)(z) = n \sum_{v=0}^{\infty} s_{n,v}(z) \int_0^{+\infty} s_{n,v}(t) f\left(\frac{nt + \alpha}{n + \beta}\right) dt,$$

where $s_{n,v}(z) = e^{-nz} \frac{(nz)^v}{v!}$.

Voronovskaja type results with quantitative estimates for these operators attached to analytic functions of exponential growth on compact disks and the exact order of approximation are found.

Throughout the present article we denote $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$. By H_R, we mean the class of all functions satisfying : $f : [R, +\infty) \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ is continuous in $(R, +\infty) \cup \overline{\mathbb{D}}_R$, analytic in $\mathbb{D}_R$ i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$.

2. Auxiliary Results

We need the following auxiliary results.

Lemma 2.1. *Let $0 \le \alpha \le \beta$, then*

$$S_n^{(\alpha,\beta)}(e_k)(z) = \sum_{j=0}^{k} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} S_n(e_j)(z),$$

where S_n denotes $S_n^{(0,0)}$.

Proof. It is immediate. $\square$

Lemma 2.2. *Let $0 \le \alpha \le \beta$ and suppose that $f : [R, +\infty) \bigcup \overline{\mathbb{D}}_R$ is analytic in $\mathbb{D}_R$ and there exists $B, C > 0$ such that $|f(x)| \le Ce^{Bx}$, for all $x \in [R, +\infty)$. Denoting $f(z) = \sum_{k=0}^{\infty} c_k z^k$, $z \in \mathbb{D}_R$, we have $S_n^{(\alpha,\beta)}(f)(z) = \sum_{k=0}^{\infty} c_k S_n^{(\alpha,\beta)}(e_k)(z)$, for all $z \in \mathbb{D}_R$.*

Proof. For any $m \in \mathbb{N}$ and $0 < r < R$, let us define

$$f_m(z) = \sum_{j=0}^{m} c_j z^j \text{ if } |z| \le r \text{ and } f_m(x) = f(x) \text{ if } x \in (r, +\infty).$$

Since $|f_m(z)| \le \sum_{j=0}^{\infty} |c_j| \cdot r^j := C_r$, for all $|z| \le r$ and $m \in \mathbb{N}$, f is continuous on $[r, R]$, from the hypothesis on f it is clear that for any $m \in \mathbb{N}$ it follows $|f_m(x)| \le C_{r,R} e^{Bx}$, for all $x \in [0, +\infty)$. This implies that for each fixed $m, n \in \mathbb{N}$, $n > B$ and z,

$$|S_n^{(\alpha,\beta)}(f_m)(z)| \le C_{r,R} |e^{-nz}| \cdot e^{B\alpha/(n+\beta)} \sum_{j=0}^{\infty} \frac{(n|z|)^j}{j!} \left(n \int_0^{\infty} \frac{n^j}{j!} t^j e^{t(Bn/(n+\beta)-n)} dt \right)$$

$$= C_{r,R} |e^{-nz}| \cdot e^{B\alpha/(n+\beta)} \sum_{j=0}^{\infty} \frac{(n|z|)^j}{j!} \cdot \frac{n^{j+1}}{(n - Bn/(n+\beta))^{j+1}} < \infty,$$

since by the ratio criterium the last series is convergent. Therefore $S_n^{(\alpha,\beta)}(f_m)(z)$ is well-defined.

Denoting

$$f_{m,k}(z) = c_k e_k(z) \text{ if } |z| \le r \text{ and } f_{m,k}(x) = \frac{f(x)}{m+1} \text{ if } x \in (r, \infty),$$

it is clear that each $f_{m,k}$ is of exponential growth on $[0, \infty)$ and that $f_m(z) = \sum_{k=0}^{m} f_{m,k}(z)$. Since from the linearity of $S_n^{(\alpha,\beta)}$ we have

$$S_n^{(\alpha,\beta)}(f_m)(z) = \sum_{k=0}^{m} c_k S_n^{(\alpha,\beta)}(e_k)(z), \text{ for all } |z| \le r,$$

it suffices to prove that $\lim_{m\to\infty} S_n^{(\alpha,\beta)}(f_m)(z) = S_n^{(\alpha,\beta)}(f)(z)$ for any fixed $n \in \mathbb{N}$ and $|z| \le r$. But this is immediate from $\lim_{m\to\infty} \|f_m - f\|_r = 0$, from $\|f_m - f\|_{B[0,+\infty)} \le \|f_m - f\|_r$ and from the inequality

$$|S_n^{(\alpha,\beta)}(f_m)(z) - S_n^{(\alpha,\beta)}(f)(z)| \le |e^{-nz}| \cdot e^{n|z|} \cdot \|f_m - f\|_{B[0,\infty)} \le M_{r,n} \|f_m - f\|_r,$$

valid for all $|z| \le r$. Here $\| \cdot \|_{B[0,+\infty)}$ denotes the uniform norm on $C[0, +\infty)$-the space of all complex-valued bounded functions on $[0, +\infty)$. $\square$

Lemma 2.3. *If we denote $S_n(e_k)(z) =: S_n^{(0,0)}(e_k)(z)$, where $e_k(z) = z^k$, then for all $|z| \leq r$ with $r \geq 1$, $n \in \mathbb{N}$ and $k = 0, 1, ..., 2...$, we have the estimate $|S_n(e_k)(z)| \leq (2k)! \cdot r^k$.*

Proof. We will use the next recurrence formula after Remark 1 in [10] :

$$S_n(e_{k+1})(z) = \frac{z}{n} S_n'(e_k)(z) + \frac{nz + k + 1}{n} S_n(e_k)(z), \ S_n(e_0)(z) = 1.$$

For $k = 0$, we get $|S_n(e_1)(z)| \leq r + \frac{1}{n}$, for all $|z| \leq r$, $n \in \mathbb{N}$. For $k = 1$, we get

$$|S_n(e_2)(z)| \leq \frac{r}{n} \cdot \|S_n'(e_1)\|_r + (r + 2/n)(r + 1/n) \leq \frac{r}{n} \cdot \frac{1}{r}(r + 1/n) + (r + 2/n)(r + 1/n)$$

$$= (r + 1/n)(r + 3/n).$$

In general, taking into account that $S_n(e_k)(z)$ is a polynomial of degree k and that by the Bernstein inequality we have $\|S_n'(e_k)(z)\| \leq \frac{k}{n} \cdot \|S_n(e_k)\|_r$, by mathematical induction we easily arrive at the inequality

$$|S_n(e_k)(z)| \leq \prod_{j=1}^{k} \left(r + \frac{2j-1}{n} \right) = r^k \prod_{j=1}^{k}(1 + (2j-1)/(nr)) \leq r^k \prod_{j=1}^{k}(1 + 2j - 1) = r^k(2k)!,$$

for all $|z| \leq r$ and $k, n \in \mathbb{N}$. $\square$

3. Main Results

Theorem 3.1. *Let $0 \leq \alpha \leq \beta$, $f \in H_R$, $1 < R < +\infty$ and suppose that there exist $M > 0$ and $A \in (\frac{1}{R}, 1)$, with the property that $|c_k| \leq M \frac{A^k}{(2k)!}$, for all $k = 0, 1, ...,$ (which implies $|f(z)| \leq Me^{A|z|}$ for all $z \in \mathbb{D}_R$) and $|f(x)| \leq Ce^{Bx}$, for all $x \in [R, +\infty)$.*

(i) Let $1 \leq r < \frac{1}{A}$. Then for all $|z| \leq r$ and $n \in \mathbb{N}$ with $n > B$, we have

$$|S_n^{(\alpha,\beta)}(f)(z) - f(z)| \leq C_{r,A} \cdot \frac{n(\beta + 1) + \beta}{n(n + \beta)},$$

where $C_{r,A} = M \cdot \sum_{k=1}^{\infty}(rA)^k < \infty$;
(ii) If $1 \leq r < r_1 < \frac{1}{A}$ are arbitrary fixed, then for all $|z| \leq r$ and $n, p \in \mathbb{N}$ with $n > B$,

$$|[S_n^{(\alpha,\beta)}]^{(p)}(f)(z) - f^{(p)}(z)| \leq \frac{p! r_1 C_{r_1,A}}{(r_1 - r)^{p+1}} \cdot \frac{n(\beta + 1) + \beta}{n(n + \beta)},$$

where $C_{r_1,A}$ is given as at the above point (i).

Proof. (i) By Lemma 2.1, we get

$$S_n^{(\alpha,\beta)}(e_k)(z) - e_k(z) = \sum_{j=0}^{k-1}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}(S_n^{(\alpha,\beta)}(e_j)(z) - e_j(z)) + \sum_{j=0}^{k-1}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}e_j(z)$$

$$+ \frac{n^k}{(n+\beta)^k}S_n(e_k)(z) - e_k(z),$$

which by using the estimate for $\|S_n(e_j) - e_j\|_r \leq \frac{(2j)!}{2n}r^{j-1}$ in the proof of Theorem 1, (i) in [10], implies

$$\|S_n^{(\alpha,\beta)}(e_k) - e_k\|_r \leq \sum_{j=0}^{k-1}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}\|S_n^{(\alpha,\beta)}(e_j) - e_j\|_r + \sum_{j=0}^{k-1}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}r^j$$

$$+ \frac{n^k}{(n+\beta)^k} \|S_n(e_k) - e_k\|_r + \left(1 - \frac{n^k}{(n+\beta)^k}\right) r^k$$

$$\leq \frac{(n+\alpha)^k}{(n+\beta)^k} \cdot \frac{(2k)!}{2n} r^{k-1} + r^k \left[\frac{(n+\alpha)^k}{(n+\beta)^k} - \frac{n^k}{(n+\beta)^k}\right] + \frac{n^k}{(n+\beta)^k} \cdot \frac{(2k)!}{2n} r^{k-1}$$

$$+ \left(1 - \frac{n^k}{(n+\beta)^k}\right) r^k \leq \frac{(2k)!}{n} r^{k-1} + 2r^k \left[1 - \frac{n^k}{(n+\beta)^k}\right] \leq \frac{(2k)!}{n} r^{k-1} + 2r^k \cdot \frac{k\beta}{n+\beta}$$

$$\leq \frac{n(\beta+1) + \beta}{n(n+\beta)} \cdot (2k)! r^k.$$

Above we upper estimated $1 - \frac{n^k}{(n+\beta)^k}$ by using the inequality $1 - \prod_{j=1}^{k} x_j \leq \sum_{j=1}^{k} (1 - x_j)$, valid for all $0 \leq x_j \leq 1$, $j = 1, ..., k$.

In conclusion, by Lemma 2.2 we can write $S_n^{(\alpha,\beta)}(f)(z) = \sum_{k=0}^{\infty} c_k S_n^{(\alpha,\beta)}(e_k)(z)$ for all $z \in \mathbb{D}_R$, $n > B$, which from the hypothesis on c_k immediately implies for all $|z| \leq r$

$$|S_n^{(\alpha,\beta)}(f)(z) - f(z)| \leq \sum_{k=1}^{\infty} |c_k| \cdot |S_n^{(\alpha,\beta)}(e_k)(z) - e_k(z)|$$

$$\leq \sum_{k=1}^{\infty} M \frac{A^k}{(2k)!} \frac{n(\beta+1) + \beta}{n(n+\beta)} \cdot (2k)! r^k$$

$$= \frac{n(\beta+1) + \beta}{n(n+\beta)} \cdot M \sum_{k=1}^{\infty} (rA)^k = C_{r,A} \cdot \frac{n(\beta+1) + \beta}{n(n+\beta)}$$

where $C_{r,A} = M \cdot \sum_{k=1}^{\infty} (rA)^k < \infty$ for all $1 \leq r < \frac{1}{A}$, taking into account that the series $\sum_{k=1}^{\infty} u^k$ is uniformly convergent in any compact disk included in the open unit disk.

(ii) Denoting by γ the circle of radius $r_1 > r$ and center 0, since for any $|z| \leq r$ and $v \in \gamma$, we have $|v - z| \geq r_1 - r$, by the Cauchy's formulas it follows that for all $|z| \leq r$ and $n \in \mathbb{N}$ with $n > B$, we have

$$|[S_n^{(\alpha,\beta)}]^{(p)}(f)(z) - f^{(p)}(z)| = \frac{p!}{2\pi} \left| \int_{\gamma} \frac{S_n^{(\alpha,\beta)}(f)(v) - f(v)}{(v-z)^{p+1}} dv \right|$$

$$\leq C_{r_1,A} \frac{n(\beta+1) + \beta}{n(n+\beta)} \cdot \frac{p!}{2\pi} \cdot \frac{2\pi r_1}{(r_1 - r)^{p+1}},$$

which proves (ii) and the theorem. $\square$

The following Voronovskaja type result holds.

Theorem 3.2. *Let $f \in H_R, 2 < R < +\infty$ and that there exist $M > 0$ and $A \in (\frac{1}{R}, 1)$, with the property that $|c_k| \leq M \frac{A^k}{(2k)!}$, for all $k = 0, 1, ...$, (which implies $|f(z)| \leq Me^{A|z|}$ for all $z \in \mathbb{D}_R$) and $|f(x)| \leq Ce^{Bx}$, for all $x \in [R, +\infty)$. If $1 \leq r < r + 1 < \frac{1}{A}$ then there exists a constant $C_{\alpha,\beta,r} > 0$ (depending only on α, β and r), such that for all $|z| \leq r$ and $n \in \mathbb{N}$ with $n > \frac{Ar}{1-Ar}$, we have*

$$\left| S_n^{(\alpha,\beta)}(f)(z) - f(z) - \frac{\alpha + 1 - \beta z}{n} f'(z) - \frac{z}{n} f''(z) \right|$$

$$\leq \frac{C_{r,A,M}(f)}{n^2} + \frac{MC_{\alpha,\beta,r}}{(n+\beta)^2} \cdot \sum_{k=0}^{\infty} k(k-1)(Ar)^k,$$

where $C_{r,A,M}(f) = \frac{2M}{r(1-Ar)} + \frac{4M}{(r+1)^2} \sum_{k=2}^{\infty} (k-1)[A(r+1)]^k < \infty$.

Proof. For all $f \in \mathbb{D}_R$, let us consider

$$S_n^{(\alpha,\beta)}(f)(z) - f(z) - \frac{\alpha + 1 - \beta z}{n} f'(z) - \frac{z}{n} f''(z)$$

$$= S_n(f)(z) - f(z) - \frac{1}{n} f'(z) - \frac{z}{n} f''(z)$$

$$+ S_n^{(\alpha,\beta)}(f)(z) - S_n(f)(z) - \frac{\alpha - \beta z}{n} f'(z).$$

Taking $f(z) = \sum_{k=0}^{\infty} c_k z^k$, we get

$$S_n^{(\alpha,\beta)}(f)(z) - f(z) - \frac{\alpha + 1 - \beta z}{n} f'(z) - \frac{z}{n} f''(z)$$

$$= \sum_{k=0}^{\infty} c_k \left(S_n(e_k, z) - z^k - \frac{k^2 z^{k-1}}{n} \right)$$

$$+ \sum_{k=0}^{\infty} c_k \left(S_n^{(\alpha,\beta)}(e_k, z) - S_n(e_k, z) - \frac{\alpha - \beta z}{n} k z^{k-1} \right).$$

By Theorem 2 of [10], for all $n > \max\{Ar/(1 - Ar), B\}$ and $|z| \le r$ we have

$$\left| \sum_{k=0}^{\infty} c_k \left(S_n(e_k, z) - z^k - \frac{k^2 z^{k-1}}{n} \right) \right| \le \frac{C_{r,A,M}(f)}{n^2},$$

where $C_{r,A,M}(f) = \frac{2M}{r(1-Ar)} + \frac{4M}{(r+1)^2} \sum_{k=2}^{\infty} (k-1)[A(r+1)]^k < \infty$. Next to estimate the second term, by using Lemma 2.1 we obtain

$$S_n^{(\alpha,\beta)}(e_k, z) - S_n(e_k, z) - \frac{\alpha - \beta z}{n} k z^{k-1}.$$

To estimate the second series we rewrite it as follows

$$S_n^{(\alpha,\beta)}(e_k, z) - S_n(e_k, z) - \frac{\alpha - \beta z}{n} k z^{k-1}$$

$$= \sum_{j=0}^{k-1} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} S_n(e_j, z) + \left(\frac{n^k}{(n+\beta)^k} - 1 \right) S_n(e_k, z) - \frac{\alpha - \beta z}{n} k z^{k-1}$$

$$= \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} S_n(e_j, z) + \frac{k n^{k-1} \alpha}{(n+\beta)^k} S_n(e_{k-1}, z)$$

$$- \sum_{j=0}^{k-1} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n+\beta)^k} S_n(e_k, z) - \frac{\alpha - \beta z}{n} k z^{k-1}$$

$$= \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} S_n(e_j, z) + \frac{k n^{k-1} \alpha}{(n+\beta)^k} \left(S_n(e_{k-1}, z) - z^{k-1} \right)$$

$$- \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n+\beta)^k} S_n(e_k, z) + \left(\frac{n^{k-1}}{(n+\beta)^{k-1}} - 1 \right) \frac{\alpha}{n+\beta} k z^{k-1}$$

$$+ \frac{k n^{k-1} \beta}{(n+\beta)^k} \left(z^k - S_n(e_k, z) \right) + \left(1 - \frac{n^{k-1}}{(n+\beta)^{k-1}} \right) \frac{\beta z}{n+\beta} k z^{k-1}$$

$$- \frac{\beta(\alpha - \beta z)}{n(n+\beta)} k z^{k-1}.$$

By using Lemma 2.3 and the inequality

$$1 - \frac{n^k}{(n+\beta)^k} \le \sum_{j=1}^{k} \left(1 - \frac{n}{n+\beta}\right) = \frac{k\beta}{n+\beta},$$

we get

$$\left|\sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} S_n\left(e_j, z\right)\right| \le \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} \left|S_n\left(e_j, z\right)\right|$$

$$= \sum_{j=0}^{k-2} \frac{(k-1)\,k}{(k-j-1)\,(k-j)} \binom{k-2}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} \left|S_n\left(e_j, z\right)\right|$$

$$\le \frac{k\,(k-1)}{2} \frac{\alpha^2}{(n+\beta)^2} r^{k-2}(2(k-2))! \sum_{j=0}^{k-2} \frac{n^j \alpha^{k-2-j}}{(n+\beta)^{k-2}}$$

$$\le \frac{k\,(k-1)}{2} \frac{\alpha^2}{(n+\beta)^2} r^{k-2}(2(k-2))!.$$

Therefore, taking into account the inequality $|S_n(e_k)(z) - z^k| \le \frac{(2k)!}{2n} r^{k-1}$, for all $k, n \in \mathbb{N}$, $|z| \le r$ in the proof of Theorem 1 in [10], it easily follows that

$$\left|S_n^{(\alpha,\beta)}(e_k, z) - S_n\left(e_k, z\right) - \frac{\alpha - \beta z}{n} k z^{k-1}\right|$$

$$\le \frac{k\,(k-1)\,\alpha^2}{2\,(n+\beta)^2} r^{k-2}(2(k-2))! + \frac{\alpha}{2(n+\beta)^2} k(2(k-1))! r^{k-2} + \frac{k\,(k-1)\,\beta^2}{2\,(n+\beta)^2} r^k(2k)!$$

$$+ \frac{k\,(k-1)\,\alpha\beta}{(n+\beta)^2} r^{k-1} + \frac{k\beta}{2(n+\beta)^2}(2k)! r^{k-1} + \frac{k\,(k-1)\,\beta}{(n+\beta)^2} r^k + \frac{\beta(1+r\beta)kr^{k-1}}{n(n+\beta)}$$

and that there exists a constant $C_{\alpha,\beta,r} > 0$ (depending only on α, β, r, which could be explicitly found by some calculation, but for simplicity we don't make it here), such that

$$\left|S_n^{(\alpha,\beta)}(e_k, z) - S_n\left(e_k, z\right) - \frac{\alpha - \beta z}{n} k z^{k-1}\right| \le \frac{C_{\alpha,\beta,r}}{(n+\beta)^2} \cdot k(k-1)r^k(2k)!.$$

In conclusion,

$$\left|\sum_{k=0}^{\infty} c_k \left(S_n^{(\alpha,\beta)}(e_k, z) - S_n(e_k, z) - \frac{\alpha - \beta z}{n} k z^{k-1}\right)\right|$$

$$\le \sum_{k=0}^{\infty} |c_k| \cdot \left|S_n^{(\alpha,\beta)}(e_k, z) - S_n(e_k, z) - \frac{\alpha - \beta z}{n} k z^{k-1}\right|$$

$$\le \frac{M C_{\alpha,\beta,r}}{(n+\beta)^2} \cdot \sum_{k=0}^{\infty} k(k-1)(Ar)^k,$$

which finally leads to the estimate in the statement. $\square$

The following exact order of approximation can be obtained.

Theorem 3.3. *(i) In the hypothesis of Theorem 3.2, if f is not a polynomial of degree ≤ 0 then for all $1 \leq r < r+1 < R$ we have*

$$\|S_n^{(\alpha,\beta)}(f) - f\|_r \sim \frac{1}{n}, \text{ for all } n > \frac{Ar}{1 - Ar},$$

where the constants in the equivalence depend only on f and r.

(ii) In the hypothesis of Theorem 3.2, if $r < r_1 < r_1 + 1 < 1/A$ and if f is not a polynomial of degree $\leq p, (p \geq 1)$ then

$$\|[S_n^{(\alpha,\beta)}]^{(p)}(f) - f^{(p)}\|_r \sim \frac{1}{n}, \text{ for all } n > \frac{Ar}{1 - Ar},$$

where the constants in the equivalence depend only on f, r, r_1 and p.

Proof. (i) For all $|z| \leq r$ and $n \in \mathbb{N}$, we can write

$$
\begin{aligned}
S_n^{(\alpha,\beta)}(f)(z) - f(z) &= \frac{1}{n}\Big[(\alpha + 1 - \beta z)f'(z) + zf''(z) \\
&\quad + \frac{1}{n} \cdot n^2 \Big(S_n^{(\alpha,\beta)}(f)(z) - f(z) - \frac{\alpha + 1 - \beta z}{n}f'(z) - \frac{z}{n}f''(z)\Big)\Big]
\end{aligned}
$$

Applying the inequality

$$\|F + G\| \geq |\,\|F\| - \|G\|\,| \geq \|F\| - \|G\|,$$

we obtain

$$\|S_n^{(\alpha,\beta)}(f) - f\|_r$$

$$\geq \frac{1}{n}\left[\left\|(\alpha + 1 - \beta e_1)f' + e_1 f''\right\|_r - \frac{1}{n} \cdot n^2 \left\|S_n^{(\alpha,\beta)}(f) - f - \frac{(\alpha + 1 - \beta e_1)f' + e_1 f''}{n}\right\|_r\right].$$

Since f is not a polynomial of degree ≤ 0 (i.e. a constant function) in $\mathbb{D}_R$, we get $\left\|(\alpha + 1 - \beta e_1)f' + e_1 f''\right\|_r > 0$. Indeed, supposing the contrary, it follows that

$$(\alpha + 1 - \beta z)f'(z) + zf''(z) = 0, \text{ for all } |z| \leq r.$$

Denoting $f'(z) = y(z)$, we get $(\alpha + 1 - \beta z)y(z) + zy'(z) = 0$, for all $|z| \leq r$. Since $y(z)$ is analytic, let $y(z) = \sum_{k=0}^{\infty} b_k z^k$. Replacing in the above equation, by the coefficients identification we easily arrive at system $(\alpha + 1)b_0 = 0, (\alpha + 1 + k)b_k = \beta b_{k-1}, k = 1, 2, ...,$.

Therefore we easily get $b_k = 0$ for all $k = 0, 1, ...,$ and $y(z) = 0$, for all $|z| \leq r$. This implies that f is a constant function, in contradiction with the hypothesis.

Now by Theorem 3.2, there exists a constant $C > 0$ independent of n, such that we have

$$n^2 \left\|S_n^{(\alpha,\beta)}(f) - f - \frac{(\alpha + 1 - \beta e_1)f' + e_1 f''}{n}\right\|_r \leq C, \text{ for all } n > \frac{Ar}{1 - Ar}.$$

Thus, there exists $n_0 > \frac{Ar}{1-Ar}$ such that for all $n \geq n_0$, we have

$$\left\|(\alpha + 1 - \beta e_1 f' + e_1 f''\right\|_r - \frac{1}{n} \cdot n^2 \left\|S_n^{(\alpha,\beta)}(f) - f - \frac{(\alpha + 1 - \beta e_1)f' + e_1 f''}{n}\right\|_r\Big]$$

$$\geq \frac{1}{2}\left\|(\alpha + 1 - \beta e_1)f' + e_1 f''\right\|_r,$$

which implies that

$$\|S_n^{(\alpha,\beta)}(f) - f\|_r \geq \frac{1}{2n}\left\|(\alpha + 1 - \beta e_1)f' + e_1 f''\right\|_r$$

for all $n \geq n_0$.

For $\frac{Ar}{1-Ar} < n \leq n_0 - 1$, we get $\|S_n^{(\alpha,\beta)}(f) - f\|_r \geq \frac{M_{r,n}(f)}{n}$ with $M_{r,n}(f) = n \cdot \|S_n^{(\alpha,\beta)}(f) - f\|_r > 0$ (since $\|S_n^{(\alpha,\beta)}(f) - f\|_r = 0$ for a certain n is valid only for f a constant function, contradicting the hypothesis on f).

Therefore, finally we have

$$\|S_n^{(\alpha,\beta)}(f) - f\|_r \geq \frac{C_r(f)}{n}$$

for all $n > \frac{Ar}{1-Ar}$, where

$$C_r(f) = \min_{n_0-1 \geq n > Ar/(1-Ar)} \left\{ M_{r,n}(f), \ldots, M_{r,n_0-1}(f), \frac{1}{2} \left\| (\alpha + 1 - \beta e_1)f' + e_1 f'' \right\|_r \right\},$$

which combined with Theorem 3.1, (i), proves the desired conclusion.

(ii) The upper estimate is exactly Theorem 3.1, (ii), therefore it remains to prove the lower estimate. Denote by Γ the circle of radius r_1 and center 0. By the Cauchy's formulas for all $|z| \leq r$ and $n \in \mathbb{N}$ we get

$$[S_n^{(\alpha,\beta)}]^{(p)}(f)(z) - f^{(p)}(z) = \frac{p!}{2\pi i} \int_\Gamma \frac{S_n^{(\alpha,\beta)}(f)(v) - f(v)}{(v-z)^{p+1}} dv,$$

where $|v - z| \geq r_1 - r$ for all $|z| \leq r$ and $v \in \Gamma$.

For all $v \in \Gamma$ and $n \in \mathbb{N}$ we get

$$S_n^{(\alpha,\beta)}(f)(v) - f(v)$$

$$= \frac{1}{n} \left\{ (\alpha + 1 - \beta v)f'(v) + v f''(v) \right.$$

$$\left. + \frac{1}{n} \left[n^2 \left(S_n^{(\alpha,\beta)}(f)(v) - f(v) - \frac{(\alpha + 1 - \beta v)f'(v) + v f''(v)}{n} \right) \right] \right\},$$

which replaced in the Cauchy's formula implies

$$[S_n^{(\alpha,\beta)}]^{(p)}(f)(z) - f^{(p)}(z) = \frac{1}{n} \left\{ \frac{p!}{2\pi i} \int_\Gamma \frac{(\alpha + 1 - \beta v)f'(v) + v f''(v)}{(v-z)^{p+1}} dv \right.$$

$$\left. + \frac{1}{n} \cdot \frac{p!}{2\pi i} \int_\Gamma \frac{n^2 \left(S_n^{(\alpha,\beta)}(f)(v) - f(v) - \frac{(\alpha+1-\beta v)f'(v)+v f''(v)}{n} \right)}{(v-z)^{p+1}} dv \right\}$$

$$= \frac{1}{n} \left\{ \left[(\alpha + 1 - \beta z)f'(z) + z f''(z) \right]^{(p)} \right.$$

$$\left. + \frac{1}{n} \cdot \frac{p!}{2\pi i} \int_\Gamma \frac{n^2 \left(S_n^{(\alpha,\beta)}(f)(v) - f(v) - \frac{(\alpha+1-\beta v)f'(v)+v f''(v)}{n} \right)}{(v-z)^{p+1}} dv \right\}.$$

Passing to the norm $\|\cdot\|_r$, for all $n \in \mathbb{N}$ we obtain

$$\|[S_n^{(\alpha,\beta)}]^{(p)}(f) - f^{(p)}\|_r$$

$$\geq \frac{1}{n} \left\{ \left\| \left[(\alpha + 1 - \beta e_1)f' + e_1 f'' \right]^{(p)} \right\|_r \right.$$

$$\left. - \frac{1}{n} \left\| \frac{p!}{2\pi} \int_\Gamma \frac{n^2 \left(S_n^{(\alpha,\beta)}(f)(v) - f(v) - \frac{(\alpha+1-\beta v)f'(v)+v f''(v)}{n} \right)}{(v-z)^{p+1}} dv \right\|_r \right\},$$

where by Theorem 3.2, for all $n > \frac{Ar}{1-Ar}$ it follows

$$\left\| \frac{p!}{2\pi} \int_\Gamma \frac{n^2 \left(S_n^{(\alpha,\beta)}(f)(v) - f(v) - \frac{(\alpha+1-\beta v)f'(v)+vf''(v)}{n} \right)}{(v-z)^{p+1}} dv \right\|_r$$

$$\leq \frac{p!}{2\pi} \cdot \frac{2\pi r_1 n^2}{(r_1-r)^{p+1}} \left\| S_n(f) - f - \frac{(\alpha+1-\beta e_1)f' + e_1 f''}{n} \right\|_{r_1} \leq C \cdot \frac{p! r_1}{(r_1-r)^{p+1}}.$$

Now, by hypothesis on f we have $\left\| [(\alpha+1-\beta e_1)f' + e_1 f'']^{(p)} \right\|_r > 0$. Indeed, supposing the contrary it follows that $(\alpha + 1 - \beta z)f'(z) + zf''(z) = Q_{p-1}(z)$ is a polynomial of degree $\leq p-1$. Denoting again $y(z) = f'(z)$ we arrive at the linear equation $(\alpha + 1 - \beta z)y(z) + zy'(z) = Q_{p-1}(z)$, with its homogenous equation having only the zero solution.

In consequence, clearly that this equation in real variable x can have as solution $y(x)$ only polynomials of degree $p-1$ in x, and from the the analyticity of y and the identity theorem, the equation in the z variable will have as solution only polynomial of degree $p-1$ in z. This implies that f necessarily is a polynomial of degree $\leq p$, in contradiction with the hypothesis.

For the rest of the proof, reasoning exactly as in the proof of the above point (i), we immediately get the required conclusion. $\square$

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